Homework assignment 8A: Solutions

1. According to the Federal Reserve System, 46% of U.S. households have credit-card debt. For a simple random sample of size $n = 30$, let $\hat{p}$ be the proportion of households with credit-card debt. What is $\mathbb{E}[\hat{p}]$ and $\sigma_{\hat{p}}$? Determine the probability that: (a) At least 10 households have credit card debt (b) At most 12 households have credit card debt.

   Answer: $p = 0.46$, $q = 0.54$, $n = 30$, $np > 5$, and $nq > 5$, so $\mathbb{E}[\hat{p}] = p = 0.46$, $\sigma_{\hat{p}} = \sqrt{\frac{0.46*0.54}{30}} = 0.091$, and $\hat{p} \sim \mathcal{N}(0.46, 0.091)$
   (a)

$$\begin{aligned}
P[\hat{p} > \frac{10}{30}] &= P[\hat{p} > 0.333] \\
&= P[\frac{\hat{p} - p}{\sigma_{\hat{p}}} > \frac{0.333 - 0.46}{0.0901}] \\
&= P[z > -1.3956] = 1 - 0.0814 = 0.9186
\end{aligned}$$

   (b)

$$\begin{aligned}
P[\hat{p} < \frac{12}{30}] &= P[\hat{p} < 0.4] \\
&= P[\frac{\hat{p} - p}{\sigma_{\hat{p}}} < \frac{0.4 - 0.46}{0.0901}] \\
&= P[z < -0.6593] = 0.2548
\end{aligned}$$

2. According to the Investment Company Institute, 40% of U.S. households have an Individual Retirement Account (IRA). Assuming the population proportion to be 0.40 and that a simple random sample of 400 households has been selected:

- What is the expected value of $\hat{p}$, the proportion in the sample having an IRA?

  Answer: $p = 0.4$, $q = 0.6$, $n = 400$, $np > 5$, and $nq > 5$, so $\mathbb{E}[\hat{p}] = p = 0.4$

- What is the standard error of the sampling distribution of the proportion?

  Answer: $\sigma_{\hat{p}} = \sqrt{\frac{0.4*0.6}{400}} = 0.0245$, and $\hat{p} \sim \mathcal{N}(0.4, 0.0245)$

- What is the probability that at least 35% of those in the sample will have an IRA?

  Answer:

$$\begin{aligned} P[\hat{p} > 0.35] &= P[\frac{\hat{p} - p}{\sigma_p} > \frac{0.35 - 0.4}{0.0245}] \\ &= P[z > -2.04] = 1 - 0.0207 = 0.9793 \end{aligned}$$

- What is the probability that between 38% and 45% of those in the sample will have an IRA?

  Answer:

$$\begin{aligned} P[0.38 < \hat{p} < 0.45] &= P[\frac{0.38 - 0.4}{0.0245} < \frac{\hat{p} - p}{\sigma_p} < \frac{0.45 - 0.4}{0.0245}] \\ &= P[-0.82 < z < 2.04] \\ &= 0.9793 - 0.2061 = 0.7732 \end{aligned}$$

3. A simple random sample with $n = 300$ is drawn from a population in which $p = 0.4$. Determine the following probabilities for $\hat{p}$, the proportion of successes in this sample: $P[\hat{p} = 0.4]$, $P[\hat{p} > 0.35]$, $P[0.38 < \hat{p} < 0.42]$, $P[\hat{p} < 0.45]$

   Answer: $p = 0.4$, $q = 0.6$, $n = 300$, $np > 5$, and $nq > 5$, so $\mathbb{E}[\hat{p}] = p = 0.46$, $\sigma_{\hat{p}} = \sqrt{\frac{0.4*0.6}{300}} = 0.0283$, and $\hat{p} \sim \mathcal{N}(0.4, 0.0283)$

$$P[\hat{p} = 0.4] = 0 \text{ , why?}$$

$$\begin{aligned} P[\hat{p} > 0.35] &= P[\frac{\hat{p} - p}{\sigma_p} > \frac{0.35 - 0.4}{0.0283}] \\ &= P[z > -1.77] = 1 - 0.0384 = 0.9616 \end{aligned}$$

$$
\begin{aligned}
P[0.38 < \hat{p} < 0.42] &= P[\frac{0.38 - 0.4}{0.0283} < \frac{\hat{p} - p}{\sigma_p} < \frac{0.42 - 0.4}{0.0283}] \\
&= P[-0.71 < z < 0.71] = \\
&= 0.7611 - 0.2389 = 0.5222
\end{aligned}
$$

$$
\begin{aligned}
P[\hat{p} < 0.45] &= P[\frac{\hat{p} - p}{\sigma_p} < \frac{0.45 - 0.4}{0.0283}] \\
&= P[z < 1.77] = 0.9616
\end{aligned}
$$

4. A population of 500 values is distributed such that $\mu = \$1000$ and $\sigma = \$400$. For a simple random sample of $n = 200$ values selected ***without replacement*** , what is is $\mathbb{E}[\bar{X}]$ and $\sigma_{\bar{X}}$?

   Answer: $n = 200$, so $\bar{X}$ is approximately normally distributed by the central limit theorem with

$$
\begin{aligned}
\mathbb{E}[\bar{X}] &= \mu = 1000 \\
\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} \times \sqrt{\frac{N-n}{N-1}} &= \frac{400}{\sqrt{200}} \times \sqrt{\frac{500-200}{500-1}} = 21.9308
\end{aligned}
$$

5. A civic organization includes 200 members, who have an average income of \$58,000, with a standard deviation of \$10,000. A simple random sample of $n = 30$ members is selected to participate in the annual fund-raising drive. What is the probability that the average income of the fund-raising group will be at least \$60,000?

   Answer: $n = 30$, so $\bar{X}$ is approximately normally distributed by the central limit theorem with

$$
\begin{aligned}
\mathbb{E}[\bar{X}] &= \mu = 58000 \\
\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} \times \sqrt{\frac{N-n}{N-1}} &= \frac{10000}{\sqrt{30}} \times \sqrt{\frac{200-30}{200-1}} = 1687.4748
\end{aligned}
$$

   In other words, $\bar{X} \sim \mathcal{N}(58000, 1687.4748)$

$$
P[\bar{X} > 60000] = P[z > 1.19] = 1 - 0.8830 = 0.1170
$$

6. A firm's receiving department has just taken in a shipment of 300 generators, 20% of which are defective. The quality control inspector has been instructed to reject the shipment if, in a simple random sample of 40, 15% or more are defective. What is the probability that the shipment will be rejected?

Answer: $p = 0.2$, $q = 0.8$, $N = 300$, $n = 40$, n is large relative to N, $np > 5$, and $nq > 5$, so $\mathbb{E}[\hat{p}] = p = 0.2$, and

$$\sigma_{\hat{p}} = \sqrt{\frac{pq}{n}} \times \sqrt{\frac{N-n}{N-1}} = \sqrt{\frac{0.2 * 0.8}{40}} \times \sqrt{\frac{300-40}{300-1}} = 0.590$$

In other words $\hat{p} \sim \mathcal{N}(0.2, 0.0590)$
A sample will be rejected whenever $\hat{p} > 0.15$, so

$$P[\hat{p} > 0.15] = P[z > \frac{0.15 - 0.2}{0.0590} = P[z > -0.85] = 0.8023$$