Homework assignment 7

1. Suppose $x \sim \mathcal{N}(20, 4)$ (That is, x is normally distributed with mean 20 and standard deviation 4), determine the following: $P[x > 20]$, $P[16 < x < 24]$, $P[x < 12]$, $P[x = 22]$, $P[12 < x < 28]$, and $P[x > 16]$.

   Answer: For all these probabilities, you need to standardize the values so that you can use the standard normal distribution table. So:

$$P[x > 20] = P[\frac{x-20}{4} > \frac{20-20}{4}] = P[z > 0] = 1 - P[z < 0] = 1 - 0.5 = 0.5$$

$$\begin{aligned} P[16 < x < 24] &= P[\frac{16-20}{4} < \frac{x-20}{4} < \frac{24-20}{4}] \\ &= P[-1 < z < 1] \\ &= P[z < 1] - P[z < -1] = 0.8413 - 0.1587 = 0.6827 \end{aligned}$$

$$P[x < 12] = P[\frac{x-20}{4} < \frac{12-20}{4}] = P[z < -2] = 0.0228$$

$$P[x = 22] = 0 \text{ because } x \text{ is continuous}$$

$$\begin{aligned} P[12 < x < 28] &= P[\frac{12-20}{4} < \frac{x-20}{4} < \frac{28-20}{4}] \\ &= P[-2 < z < 2] \\ &= P[z < 2] - P[z < -2] = 0.9772 - 0.0228 = 0.9545 \end{aligned}$$

$$P[x > 16] = P[\frac{x-20}{4} > \frac{16-20}{4}] = P[z > -1] = 1 - P[z < -1] = 1 - 0.1587 = 0.8413$$

2. Suppose $x \sim \mathcal{N}(25, 5)$ determine the following probabilities: $P[x > 25]$, $P[20 < x < 30]$, $P[x < 30]$, $P[x = 26.2]$, $P[15 < x < 25]$, and $P[x > 15]$.

   Answer: Again we need to standardize, then use the z table.

$$P[x > 25] = P[z > 0] = 1 - P[z < 0] = 1 - 0.5 = 05$$

$$P[20 < x < 30] = P[-1 < z < 1] = P[z < 1] - P[z < -1] = 0.8413 - 0.1587 = 0.6827$$

$$P[x < 30] = P[z < 1] = 0.8413$$
$$P[x = 26.2] = 0 \text{ because } x \text{ is continuous}$$
$$P[15 < x < 25] = P[-2 < z < 0] = P[z < 0] - P[z < -2] = 0.5 - 0.0228 = 0.4772$$
$$P[x > 15] = P[z > -2] = 1 - P[z < -2] = 1 - 0.0228 = 0.9772$$

3. In 2007, the average conventional first mortgage for new single-family homes was \$360,000. Assuming a normal distribution and a standard deviation of \$30,000, what proportion of the mortgages were:

   - more than \$360,000?
   - between \$300,000 and \$420,000?
   - between \$330,000 and \$390,000?
   - more than \$270,000?

   Answer: Let x be mortgage value, then $x \sim \mathcal{N}(\$360,000, \$30,000)$. Again, don't forget we need to standardize.

$$P[x > \$360,000] = P[z > 0] = 0.5$$
$$P[\$300,000 < x < 420,000] = P[-2 < z < 2] = 0.9545$$
$$P[330,000 < x < 390,000] = P[-1 < z < 1] = 0.6827$$
$$P[x > 270,000] = P[z > -3] = 1 - P[z < -3] = 0.9987$$

4. It has been reported that the average hotel check-in time, from curbside to delivery of bags into the room, is 12.0 minutes. An Li has just left the cab that brought her to her hotel. Assuming a normal distribution with a standard deviation of 2.0 minutes, what is the probability that the time required for An Li and her bags to get to the room will be:

   - greater than 14.0 minutes?
   - between 10.0 and 14.0 minutes?
   - less than 8.0 minutes?
   - between 10.0 and 16.0 minutes?

   Answer: Let x be the time it takes the bags to get delivered, then $x \sim \mathcal{N}(12, 2)$. Don't forget we need to standardize values of x so we can use the standard normal distribution table.

$$P[x > 14] = P[z > 1] = 0.1587$$
$$P[10 < x < 14] = P[-1 < z < 1] = 0.6827$$
$$P[x < 8] = P[z < -2] = 0.0228$$
$$P[10 < x < 16] = P[-1 < z < 2] = 0.8186$$

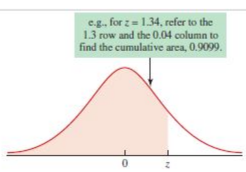
5. MENSA is an organization whose members possess IQs in the top 2% of the population. If IQs are normally distributed, with mean 100 and a standard deviation of 16, what is the minimum IQ required for admission to MENSA?

Answer: Let x be a continuous random variable representing IQ score, then $x \sim \mathcal{N}(100, 16)$. Let k be the minimum IQ score required for admission to MENSA.

Since only the top 2% qualify for MENSA membership, $P[x > k] = 0.02$. To find k we need to use the z table back wards:

$$
\begin{aligned}
P[x > k] = 0.02 &\Rightarrow P[\frac{x-\mu}{\sigma} > \frac{k-\mu}{\sigma}] = 0.02 \\
&\Rightarrow P[z > \frac{k-100}{16}] = 0.02 \\
&\Rightarrow 1 - P[z < \frac{k-100}{16}] = 0.02 \\
&\Rightarrow P[z < \frac{k-100}{16}] = 0.98
\end{aligned}
$$

Now, go to your z table (attached below) and try to find a z score which corresponds to the probability 0.98. You should notice that the z score we are looking for falls between $z = 2.05$ and $z = 2.06$, because $P[z < 2.05] = 0.9798$ and $P[z < 20.6] = 0.9803$. Here we should choose $z = 2.05$ because it is slightly closer to the probability of 0.98, but it shouldn't make a big difference if we chose $z = 2.06$ instead.



The Standard Normal Distribution

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817

Now that we have determined z, we can use it to solve for k.

$$z < \frac{k-100}{16} \Rightarrow 2.05 < \frac{k-100}{16}$$
$$\Rightarrow 2.05 * 16 < k - 100$$
$$\Rightarrow 32.8 < k - 100$$
$$\Rightarrow k > 132.8$$

Consequently, the 132.8 is the minimum IQ required for admission to MENSA.

6. Suppose the height , x, of adults in inches, is a normally distributed with mean 70. If $P[x > 79] = 0.025$ what is the standard deviation of this random normal variable?

   Answer: Again, here were are using the z table back wards to solve for σ

$$P[x > 79] = 0.025 \Rightarrow P[\frac{x-70}{\sigma} > \frac{79-70}{\sigma}] = 0.025$$
$$\Rightarrow P[z > \frac{9}{\sigma}] = 0.025$$
$$\Rightarrow 1 - P[z < \frac{9}{\sigma}] = 0.025$$
$$\Rightarrow P[z < \frac{9}{\sigma}] = 0.975$$

   Now we need to locate the z score corresponding to the probability 0.975 in the z table. From the table, you will find that $z = 1.96$.

   Now we can use $z = 1.96$ to solve for the standard deviation σ.

$$z < \frac{9}{\sigma} \Rightarrow 1.96 < \frac{9}{\sigma}$$
$$\Rightarrow 1.96\sigma < 9$$
$$\Rightarrow \sigma < \frac{9}{1.96} = 4.59$$