Homework assignment 6c

1. Suppose $x \sim \text{Bin}(25, p = 0.6)$. Using the appropriate table, determine the following: $P[x = 0]$, $P[x = 3]$, $P[x = 6]$, $P[x = 9]$, $P[x = 12]$, $P[x = 15]$, $P[x = 20]$, $P[x = 21]$, and $P[x = 25]$.

   Answer:
   $P[x = 0] = 0$, $P[x = 3] = 0$, $P[x = 6] = 0.0002$, $P[x = 9] = 0.0088$, $P[x = 12] = 0.0760$, $P[x = 15] = 0.1612$, $P[x = 20] = 0.0199$, $P[x = 21] = 0.0071$, and $P[x = 25] = 0$.

2. Suppose $x \sim \text{Bin}(25, p = 0.4)$. Using the appropriate table, determine the following: $P[x \leq 0]$, $P[x \leq 3]$, $P[x \leq 6]$, $P[x \leq 9]$, $P[x \leq 12]$, $P[x \leq 15]$, $P[x \leq 20]$, $P[x \leq 21]$, and $P[x \leq 25]$.

   Answer:
   $P[x \leq 0] = 0$, $P[x \leq 3] = 0.0024$, $P[x \leq 6] = 0.0736$, $P[x \leq 9] = 0.4246$, $P[x \leq 12] = 0.8462$, $P[x \leq 15] = 0.9868$, $P[x \leq 20] = 1$, $P[x \leq 21] = 1$, and $P[x \leq 25] = 1$.

3. Using the hypergeometric distribution, with $N = 4$, $n = 2$, and $k = 3$, determine the following: $P[x = 0]$, $P[x = 1]$, and $P[x = 2]$. What is the $\mathbb{E}[x]$? What is σ^2

   Answer: Using the formula $P[x] = \frac{C_x^k \times C_{n-x}^{N-k}}{C_n^N}$, notice that $P[x = 0]$ yields an undefined combination C_2^1. Thus, $P[x = 0] = 0$

$$P[x = 0] = \frac{C_0^3 \times C_{2-0}^{4-3}}{C_2^4} = 0$$

$$P[x = 1] = \frac{C_1^3 \times C_{2-1}^{4-3}}{C_2^4} = 0.5$$

$$P[x = 2] = \frac{C_2^3 \times C_{2-2}^{4-3}}{C_2^4} = 0.5$$

$$\mathbb{E}[x] = \frac{nk}{N} = \frac{2 \times 3}{4} = 1.5$$

$$\sigma^2 = \frac{nk(N-k)}{N^2} \times \frac{N-n}{N-1} = \frac{6}{16} \times \frac{2}{3} = 0.25$$
$$\sigma = \sqrt{0.25} = 0.5$$

4. A jar contains 10 NYC subway passes of which 6 have a balance of zero dollars. You a randomly select one card for your self, one for your friend, and an additional spare card from the jar, without replacement. What is the probability that two of the cards have a non-zero balance?

   Answer: The size of the population is $N = 10$ and the sample size is $n = 3$. If we let X be the number non-zero balance cards, then the number of success in the population is $k = 4$. So, the probability that two of the cards have a non-zero balance is:

$$P[X=2] = \frac{C_2^4 C_1^6}{C_3^{10}} = 0.3$$

5. In a criminal trial, there are 25 persons who have been approved by both parties for possible inclusion in the eventual jury of 12. Of those who have been approved, there are 14 women and 11 men. If the judge forms the final jury of 12 by randomly selecting individuals from the approved listing, what is the probability that at least half of the eventual jurors will be males?

   Answer: The size of the population is $N = 25$ and the sample size is $n = 12$. If we let x be the number of males in the jury of 12, then the number of success in the population is $k = 11$. The probability that at least half of the eventual jurors will be males is equal to $P[x \geq 6] = P[x = 6] + P[x = 7] + \cdots + P[x = 12]$.

$$P[x=6] = \frac{C_6^{11} \times C_6^{14}}{C_{12}^{25}} = 0.2668$$

$$P[x=7] = \frac{C_7^{11} \times C_5^{14}}{C_{12}^{25}} = 0.1270$$

$$P[x=8] = \frac{C_8^{11} \times C_4^{14}}{C_{12}^{25}} = 0.0318$$

$$P[x=9] = \frac{C_9^{11} \times C_3^{14}}{C_{12}^{25}} = 0.0038$$

$$P[x=10] = \frac{C_{10}^{11} \times C_2^{14}}{C_{12}^{25}} = 0.0002$$

$$P[x=11] = \frac{C_{11}^{11} \times C_1^{14}}{C_{12}^{25}} = 0.0000$$

   $P[x = 12] = 0$, because we can not have 12 males in the sample when there are only 11 males in the population.
   We can determine $P[x \geq 6]$ by adding the preceding probabilities. $P[x \geq 6] = 0.4296$

6. Unknown to a rental car office, 3 of the 12 subcompact models now available for rental are subject to a safety recall soon to be announced by the National Highway Traffic Safety Administration. Five subcompacts will be rented today, and the cars will be randomly selected from those available in the pool. What is the probability that exactly one of the recall-affected cars will be rented today? What is the probability that all three of the recall-affected cars will be rented today?

Answer: With $N = 12$, $n = 5$, and $k = 3$, the probability that exactly one of the recall-affected cars will be rented today can be calculated as:

$$P[x = 1] = \frac{C_1^3 \times C_4^9}{C_5^{12}} = 0.4773$$

The probability that all three of the recall-affected cars will be rented today can be calculated as:

$$P[x = 3] = \frac{C_3^3 \times C_2^9}{C_5^{12}} = 0.0455$$

7. Among 25 faculty, 3 have blood type O^-. Suppose we draw a simple random sample of the faculty, $n = 20$. Let the random variable x represent the number of faculty in the sample that *don't* have blood type O^-.

   a If the sample was drawn with replacement, determine $P[x = 5]$, $P[x = 10]$, $P[x = 15]$, $P[x = 18]$, $\mathbb{E}[x]$, and $Var(x)$.

   b If the sample was drawn without replacement, determine $P[x = 5]$, $P[x = 10]$, $P[x = 15]$, $P[x = 18]$, $\mathbb{E}[x]$, and $Var(x)$.

   Answer

   a This is a binomial probability distribution with $n = 20$ and $p = \frac{22}{25} = 0.88$. Since the probability of success is very high, we would expect $P[x = 5]$ and $P[x = 10]$ to be approximately zero. You can verify using the binomial function that $P[x = 15] = 0.0567$, and $P[x = 18] = 0.2740$. The expected value of x is $\mathbb{E}[x] = np = 20 * 0.88 = 17.6$, and $\sigma^2 = npq = 20 * 0.88 * 0.12 = 2.112$

   b This is a Hypergeometric distribution with $N = 25$, $n = 20$ and $k = 22$. Again, since most of the faculty don't have blood type O^-, we would expect $P[x = 5]$ and $P[x = 10]$ to be approximately zero. You can verify using the Hypergeometric distribution function that $P[x = 15] = 0$, and $P[x = 18] = 0.4130$.

$$\mathbb{E}[x] = \frac{nk}{N} = \frac{20 \times 23}{25} = 17.6$$

$$\sigma^2 = \frac{nk(N-k)}{N^2} \times \frac{N-n}{N-1} = \frac{20 * 23(25-23)}{25^2} \times \frac{25-20}{25-1} = 0.44$$

   The table below illustrates these probabilities.

	Binomial			Hypergeometric		
x	$P[x]$	$P[x] \times x$	$P[x](x - \mathbb{E}[x])^2$	$P[x]$	$P[x] \times x$	$P[x](x - \mathbb{E}[x])^2$
0	0	0	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
11	0.00021	0.00234	0.00925			
12	0.00117	0.01402	0.03663			
13	0.00527	0.06853	0.11153			
14	0.01933	0.27062	0.25047			
15	0.05670	0.85051	0.38319			
16	0.12994	2.07903	0.33250			
17	0.22421	3.81156	0.08062	0.49565	8.42609	0.17843
18	0.27403	4.93261	0.04392	0.41304	7.43478	0.06609
19	0.21153	4.01916	0.41482	0.08696	1.65217	0.17043
20	0.07756	1.55126	0.44689	0.00435	0.08696	0.02504
	0.99996	17.59964	2.10983	1.00000	17.60000	0.44000

8. For a discrete random variable that is Poisson distributed with $\lambda = 2$, determine the following: $P[x = 0]$, $P[x = 1]$, $P[x = 2]$, $P[x = 4]$, $P[x = 6]$, $P[x = 8]$, and $P[x < 3]$. What is the $\mathbb{E}[x]$? What is σ^2?

   Answer: The variance and the mean of a Poisson distribution are equal to λ, which is 2 here. You can find the probabilities in the table below. For $P[x < 3]$ you need to add $P[x = 0]$, $P[x = 1]$, and $P[x = 2]$.

x	$P[x]$
0	0.1353
1	0.2707
2	0.2707
3	0.1804
4	0.0902
5	0.0361
6	0.0120
7	0.0034
8	0.0009
9	0.0002

9. A 911 call agent receives an average of three calls an hours. If she decides to leave for lunch for an hour, what is the probability misses 0, 1, 2, 3, and 8 calls during that particular hour?

   Answer: This is a Poisson distributed random variable. Using the Poisson distribution function or a Poisson table with $\lambda = 3$ you can verify that $P[x = 0] = 0.0498$, $P[x = 1] = 0.1494$, $P[x = 2] = 0.2240$, $P[x = 3] = 0.2240$, and $P[x = 8] = 0.0081$.