Homework assignment 5

1. Machine A produces 3% defectives, machine B produces 5% defectives, and machine C produces 10% defectives. Of the total output from these machines, 60% of the items are from machine A, 30% from B, and 10% from C. One item is selected at random from a day's production for inspection. Calculate $P[D]$, the probability that the item is defective. If inspection finds the item to be defective, what is the revised probability that the item came from machine C? If inspection finds the item to be defective, what is the revised probability that the item came from machine B? If inspection finds the item to be defective, what is the revised probability that the item came from machine A?

Answer Define events: A = item is from machine A, B = item is from machine B, and C = item is from machine C. This implies that: $P[A] = 0.60; P[B] = 0.30; P[C] = 0.10; P[D|A] = 0.03; P[D|B] = 0.05; and P[D|C] = 0.1$.

$$\begin{aligned} P[D] &= P[A] * P[D|A] + P[B] * P[D|B] + P[C] * P[D|C] \\ &= [.60][.03] + [.30][.05] + [.10][.10] = 0.043 \end{aligned}$$

$$P[C|D] = \frac{P[C] * P[D|C]}{P[D]} = \frac{0.1 * 0.1}{0.043} = 0.2326$$

$$P[B|D] = \frac{P[B] * P[D|B]}{P[D]} = \frac{0.3 * 0.05}{0.043} = 0.3488$$

$$P[A|D] = \frac{P[A] * P[D|A]}{P[D]} = \frac{0.6 * 0.03}{0.043} = 0.4186$$

2. The probabilities for two events A_1 and A_2 are $P[A_1] = 0.30$ and $P[A_2] = 0.70$. It is also known that $P[A_1 \cap A_2] = 0$. Suppose $P[B|A_1] = 0.15$ and $P[B|A_2] = 0.05$. Compute $P[A_1 \cap B]$, $P[A_2 \cap B]$, and $P[B]$.

Answer Notice, that the events A_1 and A_2 are ***not*** independent because

$$P[A_1 \cap A_2] = 0 \neq P[A_1] * P[A_2] = 0.30 * 0.70 = 0.21$$

So,

$$P[A_1 \cap B] = P[A_1] * P[B|A_1] = 0.30 * 0.15 = 0.045$$
$$P[A_2 \cap B] = P[A_2] * P[B|A_2] = 0.70 * 0.05 = 0.035$$

And,

$$P[B] = P[A_1] * P[B|A_1] + P[A_2] * P[B|A_2] = 0.08$$

3. Two dice are thrown. Let E be the event that the sum of the dice is even, let F be the event that at least one of the dice lands on 6, and let G be the event that the numbers on the two dice are equal. Find $P[E]$, $P[F]$, $P[G]$, $P[E \cup F]$, $P[E \cap F]$, $P[F \cup G]$, $P[F \cap G]$.

Answer Begin by listing the sample space. This allows you to know the likelihood of all possible outcomes.

	1	2	3	4	5	6
1	$[1,1]; E \cap G$	$[1,2]$	$[1,3]; E$	$[1,4]$	$[1,5]; E$	$[1,6]; F$
2	$[2,1]$	$[2,2]; E \cap G$	$[2,3]$	$[2,4]; E$	$[2,5]$	$[2,6]; E \cap F$
3	$[3,1]; E$	$[3,2]$	$[3,3]; E \cap G$	$[3,4]$	$[3,5]; E$	$[3,6]; F$
4	$[4,1]$	$[4,2]; E$	$[4,3]$	$[4,4]; E \cap G$	$[4,5]$	$[4,6]; E \cap F$
5	$[5,1]; E$	$[5,2]$	$[5,3]; E$	$[5,4]$	$[5,5]; E \cap G$	$[5,6]; F$
6	$[6,1]; F$	$[6,2]; E \cap F$	$[6,3]; F$	$[6,4]; E \cap F$	$[6,5]; F$	$[6,6]; E \cap F \cap G$

Hopefully, now you can see that:

$$P[E] = \frac{18}{36} \qquad P[G] = \frac{6}{36} \qquad P[F \cap G] = \frac{1}{36}$$
$$P[E \cap F] = \frac{5}{36}$$
$$P[F] = \frac{11}{36} \qquad P[E \cup F] = \frac{24}{36} \qquad P[F \cup G] = \frac{16}{36}$$

4. Suppose we draw a card from a shuffled set of 52 playing cards. What is the probability of drawing a Queen, given that the card drawn is of suit Hearts $\heartsuit$?

Answer

$$P[Q|\heartsuit] = \frac{P[Q \cap \heartsuit]}{P[\heartsuit]} = \frac{\frac{1}{52}}{\frac{13}{52}} = \frac{1}{13}$$

5. A motorcycle insurance company classifies riders as risky or safe. Within a given year, a risky rider will have an accident with probability 0.5. That is, $P[accident|risky] = 0.5$. This probability decreases to 0.1 for a safe rider. Assume that 20% of riders are risky. Given that a rider has just had an accident, what is the probability that he/she is risky.

Answer We are given the following:
The probability that a rider is risky $P[risky] = 0.2$. So, the probability that a rider is safe is $P[safe] = 0.8$

We are also given that: $P[accident|risky] = 0.5$ and $P[accident|safe] = 0.1$

We are asked to find $P[risky|accident]$ which can be expressed as

$$P[risky|accident] = \frac{P[risky \cap accident]}{P[accident]}$$

From $P[accident|risky] = 0.5$, we know that

$$P[accident|risky] = \frac{P[accident \cap risky]}{P[risky]} = \frac{P[accident \cap risky]}{0.2} = 0.5$$

This implies that

$$P[accident \cap risky] = 0.5 * 0.2 = 0.1$$

Similarly, from $P[accident|safe] = 0.1$, we know that

$$P[accident|safe] = \frac{P[accident \cap safe]}{P[safe]} = \frac{P[accident \cap safe]}{0.8} = 0.1$$

This implies that

$$P[accident \cap safe] = 0.8 * 0.1 = 0.08$$

We can now calculate the probability of an accident using the law of total probability:

$$P[accident] = P[accident \cap safe] + P[accident \cap risky] = 0.08 + 0.1 = 0.18$$

Finally, we can use these values to calculate $P[risky|accident]$:

$$P[risky|accident] = \frac{P[risky \cap accident]}{P[accident]} = \frac{0.1}{0.18} = 0.5555556$$

You might find the table below helpful:

	accident	No accident	
risky	0.1	0.1	0.2
safe	0.1	0.62	0.8
	0.18	0.72	1

6. An investment counselor would like to meet with 14 of his clients on Wednesday, but he only has time for 10 appointments. How many different combinations of the clients could be considered for inclusion into his limited schedule for that day?

Answer Notice that this is not a permutation since the order of the meetings doesn't matter. So, the number of ways the counselor can meet a set of 10 clients out of 14 is

$$\begin{aligned} C_{10}^{14} &= \frac{14!}{10!(14-10)!} \\ &= \frac{14 * 13 * \cancel{12} * 11 * \cancel{10!}}{\cancel{10!} * \cancel{4 * 3} * 2} \\ &= \frac{\cancel{14}^{7*\cancel{2}} * 13 * 11}{\cancel{2}} \\ &= 7 * 13 * 11 = 1001 \end{aligned}$$

7. How many different combinations are possible if 8 substitute workers are available to fill 4 openings created by employees planning to take vacation leave next week?

Answer Again, notice that this is not a permutation since order doesn't matter. So, the number of possible ways for 8 substitute workers to fill in 4 openings is:

$$\begin{aligned} C_4^8 &= \frac{8!}{4!(8-4)!} \\ &= \frac{8*7*\cancel{6}*5*\cancel{4!}}{\cancel{4!}*4*\cancel{3*2}} \\ &= \frac{\overset{2*\cancel{4}}{\cancel{8}}*7*5}{\cancel{4}} \\ &= 2*7*5 = 70 \end{aligned}$$

8. Ten students from an economics class have formed a study group. Each may or may not attend a study session. Assuming that the members will be making independent decisions on whether or not to attend, how many different possibilities exist for the composition of the study session?

Answer Once more, notice that this is not a permutation since order doesn't matter. We are only interested in the number of ways the group of attendees can be made up. If only one student attends, we have C_1^{10} ways to make up the group; if 2 students attend, we have C_2^{10} ways to make up the group; if 3 students attend, we have C_3^{10} ways to make up the group; and so on. It's also possible that no one attends. So, the total number of possible ways for the composition of the study session is:

$$\begin{aligned} \sum_{r=0}^{10} C_r^{10} &= C_0^{10} + C_1^{10} + C_2^{10} + C_3^{10} + C_4^{10} + C_5^{10} + C_6^{10} + C_7^{10} + C_8^{10} + C_9^{10} + C_{10}^{10} \\ &= 1 + 10 + 45 + 120 + 210 + 252 + 210 + 120 + 45 + 10 + 1 \\ &= 1024 \end{aligned}$$

9. A state's license plate has 9 positions, each of which has 37 possibilities (26 letters, 10 integers, or blank). If the purchaser of a vanity plate wants his first seven positions to be Stanley, in how many ways can his plate appear?

Answer In this case the first seven positions will be taken by S-t-a-n-l-e-y leaving us with two positions to fill. These two remaining positions can be filled with anything from the 37 possibilities (26 letters, 10 integers, or blank). Notice that order matters, and that repetition is allowed. So, we have permutations with repetition or $PR(n, r) = n^r$ where $n = 37$ and $r = 2$.

$$PR(37, 2) = 37^2 = 1369$$