Chapter 5: Probability; Review of Basic Concepts

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September 25, 2017

Probability: Basic Terms

Definitions

An ***experiment*** is an activity or measurement that results in an outcome.

A ***sample space*** is the set of all possible outcomes of an experiment.

An ***event*** is one or more of the possible outcomes of an experiment; it's a subset of the sample space.

A ***probability*** is a number between 0 and 1 that expresses the chance that an event will occur

Basic probability terms: an example

Example

Experiment: role a six-sided die once.

Sample Space: $\{1, 2, 3, 4, 5, 6\}$

Events:

$A_1 = \{2, 4, 6\}$: Rolling an even number

$A_2 = \{1, 3, 5\}$: Rolling an odd number

$A_3 = \{6\}$: Rolling a six.

Probabilities:

$P[A_1]$ = Probability of rolling an even number

$P[A_2]$ = Probability of rolling an odd number

$P[A_3]$ = Probability of rolling a six

Basic probability terms: an example

Example

Experiment: flip a double sided coin.

Sample Space: $\{H, T\}$

Events:

$A = \{H\}$: The coin lands on Heads

$B = \{T\}$: : The coin land on Tails

Probabilities:

$P[A] =$ Probability that the coin lands on Heads

$P[A']$ = Probability that the coin The coin land on Tails

Probability: Basic Terms

Definition

Events are ***mutually exclusive*** if, when one event occurs, the other cannot occur.

A set of events is ***exhaustive*** if it includes all the possible outcomes of an experiment. In other words, it includes all elements s_i in the sample space S.

The ***complement*** of an event A, denoted $A^{'}$, is the event not occurring. The event A and its complement $A^{'}$ are *mutually exclusive* and *exhaustive*.

What's a probability?

The Classical Approach

For outcomes that are equally likely,

$$\text{Probability} = \frac{\text{Number of possible outcomes in which the event occurs}}{\text{Total number of possible outcomes}}$$

Example

role a six-sided die once.

$A_1 = \{2, 4, 6\}$: Rolling an even number. $P[A_1] = \frac{3}{6}$.

$A_2 = \{1, 3, 5\}$: Rolling an odd number. $P[A_2] = \frac{3}{6}$.

$A_3 = \{6\}$: Rolling a six. $P[A_3] = \frac{1}{6}$.

What's a probability?

The Relative Frequency Approach

Probability is the *proportion of times an event is observed to occur in a very large number of trials*:

$$\text{Probability} = \frac{\text{Number of trials in which the event occurs}}{\text{Total number of trials}}$$

Law of Large Numbers

Over a large number of trials, the relative frequency with which an event occurs will approach the probability of its occurrence for a single trial.

The Classical Probability Approach: Example

Suppose we roll two fair six-sided dice, and add them up. What is the probability of each sum? And what is the most likely sum?

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Suppose we roll two fair six-sided dice, and add them up. What is the probability of each sum? And what is the most likely sum?

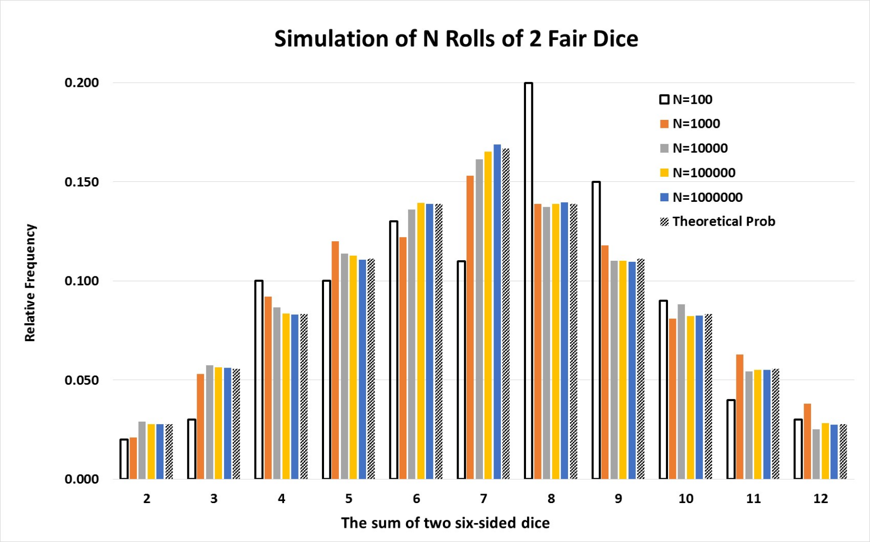
Sum	N. of outcomes	Outcomes	Probability
2	1	{1 1}	$1/36 = 0.028$
3	2	{1 2 , 2 1}	$2/36 = 0.056$
4	3	{1 3, 2 2, 3 1}	$3/36 = 0.083$
5	4	{1 4, 2 3, 3 2, 4 1}	$4/36 = 0.111$
6	5	{1 5, 2 4, 3 3, 4 2, 5 1}	$5/36 = 0.139$
7	6	{1 6, 2 5, 3 4, 4 3, 5 2, 6 1}	$6/36 = 0.167$
8	5	{2 6, 3 5, 4 4 , 5 3, 6 2}	$5/36 = 0.139$
9	4	{3 6, 4 5, 5 4, 6 3}	$4/36 = 0.111$
10	3	{4 6, 5 5, 6 4}	$3/36 = 0.083$
11	2	{5 6, 6 5}	$2/36 = 0.056$
12	1	{6 6}	$1/36 = 0.028$
	36		$36/36 = 1.00$

The Relative Frequency Approach: Example

Suppose we roll two fair six-sided dice, and add them up. Suppose we do it 100s, 1000s and millions of times.

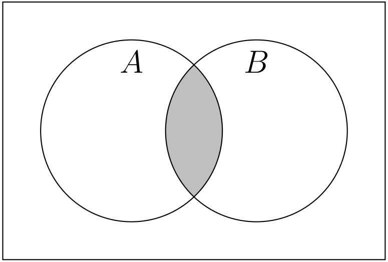
Sum	N=100	N=1000	N=10000	N=100000	N=1000000	Theoretical Probability
2	0.020	0.021	0.029	0.028	0.028	0.028
3	0.030	0.053	0.058	0.056	0.056	0.056
4	0.100	0.092	0.087	0.083	0.083	0.083
5	0.100	0.120	0.114	0.113	0.111	0.111
6	0.130	0.122	0.136	0.139	0.139	0.139
7	0.110	0.153	0.161	0.165	0.169	0.167
8	0.200	0.139	0.137	0.139	0.140	0.139
9	0.150	0.118	0.110	0.110	0.110	0.111
10	0.090	0.081	0.088	0.082	0.083	0.083
11	0.040	0.063	0.055	0.055	0.055	0.056
12	0.030	0.038	0.025	0.028	0.027	0.028
	1	1	1	1	1	1

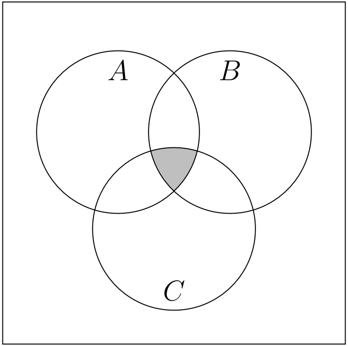
Relative frequency of the sum of 2 fair dice.



Intersection of events

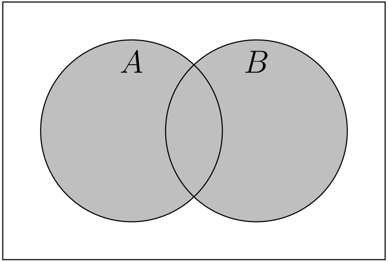
Two or more events ***intersect*** if they occur at the same time. Such an intersection can be represented by $A \cap B$ for "A and B," or $A \cap B \cap C$ for "A and B and C," depending on the number of possible events involved.

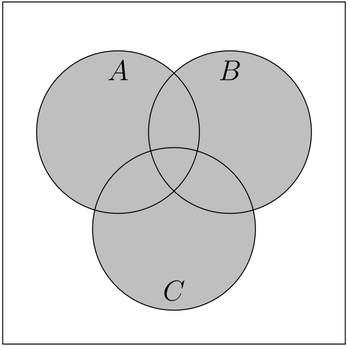




Union of events

The ***union*** of two or more events is the set of elements which belong to at least one of the events. The union can be represented by $A \cup B$ for "A or B," or $A \cup B \cup C$ for "A or B or C," depending on the number of possible events involved.





In a certain residential suburb 60% of all households subscribe to the metropolitan newspaper, 77% subscribe to the local paper, and 44% to both newspapers. What proportion of households subscribe to exactly one of the two newspapers?

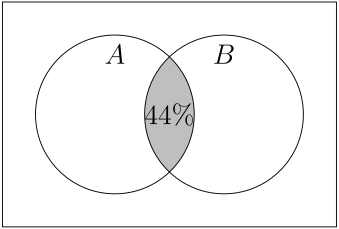
Answer:

Unions and Intersections of Events: Example 1

In a certain residential suburb 60% of all households subscribe to the metropolitan newspaper, 77% subscribe to the local paper, and 44% to both newspapers. What proportion of households subscribe to exactly one of the two newspapers?

Answer:

Let:
A = metropolitan subscribers
B = local paper subscribers
Since, $A \cap B = 44\%$, we have
$A \cap B' = 60 - 44 = 16\%$
$B \cap A' = 77 - 44 = 33\%$.



Unions and Intersections of Events: Example 2

The following data shows frequencies describing the sex and age of persons injured by fireworks in 1995. Let event A represent males, and event B represent individuals under 15 of age. What are $A \cap B$,$A \cap B'$, $A' \cap B$,$A' \cup B'$,$A' \cap B'$ and $A \cup B$?

		Age		
		B Under 15	***B′*** 15 or Older	
A	**Male**	3477	5436	8913
A′	**Female**	1249	1287	2536
		4726	6723	11,449

Probability Basic Axioms

Axioms

Let A be an event in the sample space S. The following axioms always hold:

- $0 \leq P[A] \leq 1$
- $\sum_{i=1}^{n} P_i[s_i] = 1 \; \forall \; s_i \in S$.
- $P[A] + P[A'] = 1$
- $P[A] = 0 \implies A \notin S$, A is impossible
- $P[A] = 1 \implies A$ is certain

Joint and Conditional Probabilities

Definitions

Marginal Probability: The probability that a given event will occur. No other events are taken into consideration. A typical expression is $P[A]$.

Joint Probability: The probability that two or more events will all occur. Usually expressed as $P[A \cap B]$, $P[A' \cup B]$.

Conditional Probability: The probability that an event will occur, given that another event has already happened. We denote the probability of A, given B as $P[A|B]$

Conditional Probability, Example 1

A bag contains 850 marbles, of which 50 are red and 60 are blue. One marble is selected at random, with replacement. Let A denote the event that the 1^{st} marble selected is red, and B denotes the event that the 2^{nd} marble selected is blue. Find $P[A]$, $P[B]$, and $P[B|A]$.

Answer:

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A bag contains 850 marbles, of which 50 are red and 60 are blue. One marble is selected at random, with replacement. Let A denote the event that the 1^{st} marble selected is red, and B denotes the event that the 2^{nd} marble selected is blue. Find $P[A]$, $P[B]$, and $P[B|A]$.

Answer:
$P[A] = \frac{50}{850}$

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Answer:

$P[A] = \frac{50}{850}$

$P[B] = \frac{60}{850}$

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Answer:

$P[A] = \frac{50}{850}$

$P[B] = \frac{60}{850}$

$P[B|A] = P[B] = \frac{60}{850}$

Conditional Probability, Example 1

A bag contains 850 marbles, of which 50 are red and 60 are blue. One marble is selected at random, with replacement. Let A denote the event that the 1^{st} marble selected is red, and B denotes the event that the 2^{nd} marble selected is blue. Find $P[A]$, $P[B]$, and $P[B|A]$.

Answer:
$P[A] = \frac{50}{850}$
$P[B] = \frac{60}{850}$
$P[B|A] = P[B] = \frac{60}{850}$
Now suppose the two marbles are selected without replacement. Find $P[B|A]$.

Answer:

Conditional Probability, Example 1

A bag contains 850 marbles, of which 50 are red and 60 are blue. One marble is selected at random, with replacement. Let A denote the event that the 1^{st} marble selected is red, and B denotes the event that the 2^{nd} marble selected is blue. Find $P[A]$, $P[B]$, and $P[B|A]$.

Answer:
$P[A] = \frac{50}{850}$
$P[B] = \frac{60}{850}$
$P[B|A] = P[B] = \frac{60}{850}$
Now suppose the two marbles are selected without replacement. Find $P[B|A]$.

Answer:
$P[B|A] = \frac{60}{849} \neq P[B]$

Independence

Definition

Two events are ***independent*** when the occurrence of one event has no effect on the probability that another will occur.
Events are ***dependent*** when the occurrence of one event changes the probability that another will occur.

Example

Suppose we toss a coin twice. The sample space is $S = \{HH, HT, TH, TT\}$. Let event A represent getting heads in the first toss, and let B represent getting tails in the second toss. We say that A and B are independent, because the realization of any of them doesn't affect the realization of the other.

Rules of probability

- $P[A \cup B] = P[A] + P[B] - P[A \cap B]$
- $P[A \cup B] = P[A] + P[B]$ When events A and B are are mutually exclusive
- $P[A \cap B] = P[A] \times P[B]$ when events A and B are independent
- The probability of A conditional on B is:

$$P[A|B] = \frac{P[A \cap B]}{P[B]}, \forall \text{ B such that } P[B] \neq 0$$

- When A and B are independent:

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[A] \times P[B]}{P[B]} = P[A]$$

- $P[A \cap B] = P[A] \times P[B|A]$ when events A and B are not independent

Example: Marginal Probabilities

From Table 1, the marginal probabilities are:

Table 1: Prob. of injury by fireworks

		Age		
		$B :< 15$	$B' :\geq 15$	
Sex	A:Male	0.304	0.475	0.779
	A':Female	0.109	0.112	0.221
		0.413	0.587	

- $P[A] =$

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- $P[A] = 0.779$
- $P[B] =$

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- $P[A] = 0.779$
- $P[B] = 0.413$
- $P[A'] =$

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- $P[A] = 0.779$
- $P[B] = 0.413$
- $P[A'] = 1 - P[A] = 0.221$
- $P[B'] =$

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- $P[A] = 0.779$
- $P[B] = 0.413$
- $P[A'] = 1 - P[A] = 0.221$
- $P[B'] = 1 - P[B] = 0.587$

Joint Probabilities: Example

From Table 2, the joint probabilities are:

Table 2: Prob. of injury by fireworks

		Age		
		$B :< 15$	$B' :\geq 15$	
Sex	A:Male	0.304	0.475	0.779
	A':Female	0.109	0.112	0.221
		0.413	0.587	

- $P[A \cap B] =$

Joint Probabilities: Example

From Table 2, the joint probabilities are:

Table 2: Prob. of injury by fireworks

		Age: $B :< 15$	Age: $B' :\geq 15$	
Sex	A:Male	0.304	0.475	0.779
	A':Female	0.109	0.112	0.221
		0.413	0.587	

- $P[A \cap B] = 0.304$
- $P[A \cap B'] =$

Joint Probabilities: Example

From Table 2, the joint probabilities are:

Table 2: Prob. of injury by fireworks

		Age		
		$B :< 15$	$B' :\geq 15$	
Sex	A:Male	0.304	0.475	0.779
	A':Female	0.109	0.112	0.221
		0.413	0.587	

- $P[A \cap B] = 0.304$
- $P[A \cap B'] = 0.475$
- $P[A' \cap B] =$

Joint Probabilities: Example

From Table 2, the joint probabilities are:

Table 2: Prob. of injury by fireworks

		Age $B :< 15$	Age $B' :\geq 15$	
Sex	A:Male	0.304	0.475	0.779
	A':Female	0.109	0.112	0.221
		0.413	0.587	

- $P[A \cap B] = 0.304$
- $P[A \cap B'] = 0.475$
- $P[A' \cap B] = 0.109$
- $P[A' \cap B'] =$

Joint Probabilities: Example

From Table 2, the joint probabilities are:

Table 2: Prob. of injury by fireworks

		Age: $B :< 15$	Age: $B' :\geq 15$	
Sex	A:Male	0.304	0.475	0.779
	A':Female	0.109	0.112	0.221
		0.413	0.587	

- $P[A \cap B] = 0.304$
- $P[A \cap B'] = 0.475$
- $P[A' \cap B] = 0.109$
- $P[A' \cap B'] = 0.112$

Joint Probabilities: Example Contd.

From Table 3, the joint probabilities are:

Table 3: Prob. of injury by fireworks

		Age		
		$B :< 15$	$B' :\geq 15$	
Sex	A:Male	0.304	0.475	0.779
	A':Female	0.109	0.112	0.221
		0.413	0.587	

$P[A \cup B] =$

Joint Probabilities: Example Contd.

From Table 3, the joint probabilities are:

Table 3: Prob. of injury by fireworks

		Age		
		$B :< 15$	$B' :\geq 15$	
Sex	A:Male	0.304	0.475	0.779
	A':Female	0.109	0.112	0.221
		0.413	0.587	

$P[A \cup B] =$
$0.779 + 0.413 - 0.304 = 0.888$

$P[A \cup B'] =$

Joint Probabilities: Example Contd.

From Table 3, the joint probabilities are:

Table 3: Prob. of injury by fireworks

		Age		
		$B :< 15$	$B' :\geq 15$	
Sex	A:Male	0.304	0.475	0.779
	A':Female	0.109	0.112	0.221
		0.413	0.587	

$P[A \cup B] =$
$0.779 + 0.413 - 0.304 = 0.888$

$P[A \cup B'] =$
$0.779 + 0.587 - 0.475 = 0.891$

$P[A' \cup B] =$

Joint Probabilities: Example Contd.

From Table 3, the joint probabilities are:

Table 3: Prob. of injury by fireworks

		Age		
		$B :< 15$	$B' :\geq 15$	
Sex	A:Male	0.304	0.475	0.779
	A':Female	0.109	0.112	0.221
		0.413	0.587	

$P[A \cup B] =$
$0.779 + 0.413 - 0.304 = 0.888$

$P[A \cup B'] =$
$0.779 + 0.587 - 0.475 = 0.891$

$P[A' \cup B] =$
$0.221 + 0.413 - 0.109 = 0.525$

$P[A' \cup B'] =$

Joint Probabilities: Example Contd.

From Table 3, the joint probabilities are:

Table 3: Prob. of injury by fireworks

		Age		
		$B :< 15$	$B' :\geq 15$	
Sex	A:Male	0.304	0.475	0.779
	A':Female	0.109	0.112	0.221
		0.413	0.587	

$P[A \cup B] =$
$0.779 + 0.413 - 0.304 = 0.888$

$P[A \cup B'] =$
$0.779 + 0.587 - 0.475 = 0.891$

$P[A' \cup B] =$
$0.221 + 0.413 - 0.109 = 0.525$

$P[A' \cup B'] =$
$0.221 + 0.587 - 0.112 = 0.696$

Conditional Probabilities: Example

$$P[A|B] =$$

Conditional Probabilities: Example

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{0.304}{0.413} = 0.736$$

$$P[B|A] =$$

Conditional Probabilities: Example

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{0.304}{0.413} = 0.736$$

$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{0.304}{0.779} = 0.390$$

$$P[B'|A] =$$

Conditional Probabilities: Example

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{0.304}{0.413} = 0.736$$

$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{0.304}{0.779} = 0.390$$

$$P[B'|A] = \frac{P[A \cap B']}{P[A]} = \frac{0.475}{0.779} = 0.610$$

$$P[A'|B] =$$

Conditional Probabilities: Example

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{0.304}{0.413} = 0.736$$

$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{0.304}{0.779} = 0.390$$

$$P[B'|A] = \frac{P[A \cap B']}{P[A]} = \frac{0.475}{0.779} = 0.610$$

$$P[A'|B] = \frac{P[A' \cap B]}{P[B]} = \frac{0.109}{0.413} = 0.264$$

$$P[A'|B'] =$$

Conditional Probabilities: Example

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{0.304}{0.413} = 0.736$$

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$$P[B'|A] = \frac{P[A \cap B']}{P[A]} = \frac{0.475}{0.779} = 0.610$$

$$P[A'|B] = \frac{P[A' \cap B]}{P[B]} = \frac{0.109}{0.413} = 0.264$$

$$P[A'|B'] = \frac{P[A' \cap B']}{P[B']} = \frac{0.112}{0.587} = 0.191$$

Practice Problem 1

A financial adviser frequently holds investment counseling workshops for persons who have responded to his direct mailings. The typical workshop has 10 attendees. In the past, the adviser has found that in 35% of the workshops, nobody signs up for the advanced class that is offered; in 30% of the workshops, one person signs up; in 25% of the workshops, two people sign up; and in 10% of the workshops, three or more people sign up. The adviser is holding a workshop tomorrow. What is the probability that at least two people will sign up for the advanced class? What is the probability that no more than one person will sign up?

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A financial adviser frequently holds investment counseling workshops for persons who have responded to his direct mailings. The typical workshop has 10 attendees. In the past, the adviser has found that in 35% of the workshops, nobody signs up for the advanced class that is offered; in 30% of the workshops, one person signs up; in 25% of the workshops, two people sign up; and in 10% of the workshops, three or more people sign up. The adviser is holding a workshop tomorrow. What is the probability that at least two people will sign up for the advanced class? What is the probability that no more than one person will sign up?

Answer:

Probability that at least two people will sign up is $0.25 + 0.10 = 0.35$;

Probability that no more than one person signs up is $0.35 + 0.30 = 0.65$.

Practice Problem 2

For three mutually exclusive events, $P[A] = 0.3$, $P[B] = 0.6$, and $P[A \cup B \cup C] = 1$. What is the value of $P[A \cup C]$?

Answer:

Practice Problem 2

For three mutually exclusive events, $P[A] = 0.3$, $P[B] = 0.6$, and $P[A \cup B \cup C] = 1$. What is the value of $P[A \cup C]$?

Answer:

$P[A \cup C] = P[A] + P[C] - P[A \cap C]$. We know that $P[A \cap C] = 0$ because the events A and C are mutually exclusive. So, we just need to find $P[C]$.

Since A, B, and C are mutually exclusive,
$P[A \cup B \cup C] = P[A] + P[B] + P[C] = 1$. So,
$P[C] = 1 - P[A] - P[B] = 0.1$.

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For three mutually exclusive events, $P[A] = 0.3$, $P[B] = 0.6$, and $P[A \cup B \cup C] = 1$. What is the value of $P[A \cup C]$?

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Since A, B, and C are mutually exclusive,
$P[A \cup B \cup C] = P[A] + P[B] + P[C] = 1$. So,
$P[C] = 1 - P[A] - P[B] = 0.1$.

And, $P[A \cup C] = P[A] + P[C] = 0.3 + 0.1 = 0.4$

Practice Problem 3

It has been reported that 57% of U.S. households that rent do not have a dishwasher, while only 28% of homeowner households do not have a dishwasher. If one household is randomly selected from each ownership category, determine the probability that:

- neither household will have a dishwasher.
- both households will have a dishwasher.
- the renter household will have a dishwasher, but the homeowner household will not.
- the homeowner household will have a dishwasher, but the renter household will not.

Hint: Let A = The renter has a dishwasher, and B = The homeowner has a dishwasher. Then, $P[A'] = 0.57$ and $P[B'] = 0.28$. So, $P[A] = 1 - 0.57 = 0.43$ and $P[B] = 1 - 0.28 = 0.72$.

Practice Problem 3

Answer: Recall, A = Renter household has a dishwasher. B = Homeowner household has a dishwasher. We are told $P[A'] = 0.57$ and $P[B'] = 0.28$. Therefore, $P[A] = 1 - 0.57 = 0.43$ and $P[B] = 1 - 0.28 = 0.72$. Also, notice that the events A and B are independent.

- neither household will have a dishwasher:
  $P[A' \cap B'] = P[A'] * P[B'] = 0.57 * 0.28 = 0.16$

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- both households will have a dishwasher:
  $P[A \cap B] = P[A] * P[B] = 0.43 * 0.72 = 0.31$

Practice Problem 3

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  $P[A \cap B] = P[A] * P[B] = 0.43 * 0.72 = 0.31$
- the renter household will have a dishwasher, but the homeowner household will not: $P[A \cap B'] = P[A] * P[B'] = 0.43 * 0.28 = 0.12$

Practice Problem 3

Answer: Recall, A = Renter household has a dishwasher. B = Homeowner household has a dishwasher. We are told $P[A'] = 0.57$ and $P[B'] = 0.28$. Therefore, $P[A] = 1 - 0.57 = 0.43$ and $P[B] = 1 - 0.28 = 0.72$. Also, notice that the events A and B are independent.

- neither household will have a dishwasher: $P[A' \cap B'] = P[A'] * P[B'] = 0.57 * 0.28 = 0.16$
- both households will have a dishwasher: $P[A \cap B] = P[A] * P[B] = 0.43 * 0.72 = 0.31$
- the renter household will have a dishwasher, but the homeowner household will not: $P[A \cap B'] = P[A] * P[B'] = 0.43 * 0.28 = 0.12$
- the homeowner household will have a dishwasher, but the renter household will not: $P[B \cap A'] = P[B] * P[A'] = 0.72 * 0.57 = 0.41$

Practice Problem 3

Answer: Recall, A = Renter household has a dishwasher. B = Homeowner household has a dishwasher. We are told $P[A'] = 0.57$ and $P[B'] = 0.28$. Therefore, $P[A] = 1 - 0.57 = 0.43$ and $P[B] = 1 - 0.28 = 0.72$. Also, notice that the events A and B are independent.

- neither household will have a dishwasher: $P[A' \cap B'] = P[A'] * P[B'] = 0.57 * 0.28 = 0.16$
- both households will have a dishwasher: $P[A \cap B] = P[A] * P[B] = 0.43 * 0.72 = 0.31$
- the renter household will have a dishwasher, but the homeowner household will not: $P[A \cap B'] = P[A] * P[B'] = 0.43 * 0.28 = 0.12$
- the homeowner household will have a dishwasher, but the renter household will not: $P[B \cap A'] = P[B] * P[A'] = 0.72 * 0.57 = 0.41$

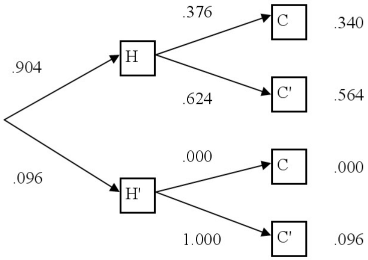
Practice Problem 4

Of employed U.S. adults age 25 or older, 90.4% have completed high school, while 34.0% have completed college. For H = completed high school, C = completed college, and assuming that one must complete high school before completing college, construct a tree diagram to assist your calculation of the following probabilities for an employed U.S. adult:

- $P[H]$
- $P[H \cap C]$
- $P[C|H]$
- $P[H \cap C']$

Practice Problem 4

Answer

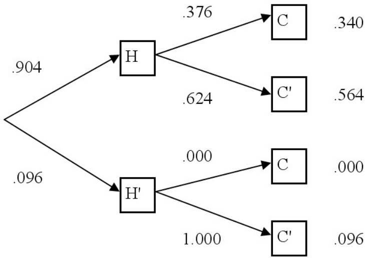


From the statement of the problem, we know $P[H] = 0.904$ and $P[C] = P[H \cap C] = 0.340$.

- $P[C|H] =$

Practice Problem 4

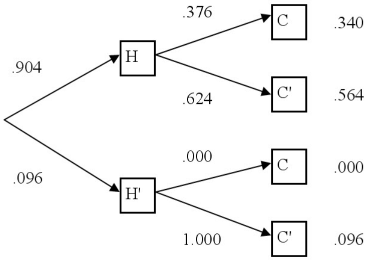
Answer



From the statement of the problem, we know $P[H] = 0.904$ and $P[C] = P[H \cap C] = 0.340$.

- $P[C|H] = \frac{P[C \cap H]}{P[H]} = 0.340/0.904 = 0.376$
- $P[H \cap C'] =$

Answer



From the statement of the problem, we know $P[H] = 0.904$ and $P[C] = P[H \cap C] = 0.340$.

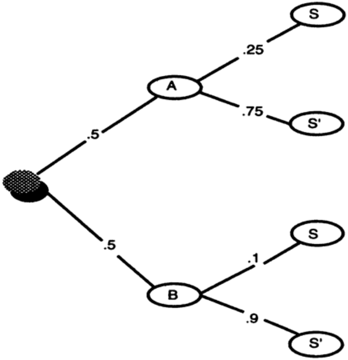
- $P[C|H] = \frac{P[C \cap H]}{P[H]} = 0.340/0.904 = 0.376$
- $P[H \cap C'] = P[H] * P[C'|H] = 0.904 * 0.624 = 0.564$

Practice Problem 5

A taxi company in a small town has two cabs. Cab A stalls at a red light 25% of the time, while cab B stalls just 10% of the time. A driver randomly selects one of the cars for the first trip of the day. What is the probability that the engine will stall at the first red light the driver encounters?

Practice Problem 5

Answer Define the following events: A = driver takes cab A, B = driver takes cab B, S = cab stalls at the light. We know $P[A] = 0.5$, $P[B] = 0.5$, $P[S|A] = 0.25$, and $P[S|B] = 0.1$. See the tree diagram below.



$P[S] = P[A \cap S] + P[B \cap S] = P[A] * P[S|A] + P[B] * P[S|B] =$
$0.5 * 0.25 + 0.5 * 0.1 = 0.175$

More probability examples

Suppose we have 3 shirts $\{s_1, s_2, s_3\}$ and 2 pairs of paints $\{p_1, p_2\}$ (a typical graduate student wardrobe). Suppose the owner is equally likely to wear any combination of pants and any of the shirts. What are the following probabilities:

- $P[p_1]$
- $P[p_2]$
- $P[p_1 \cap s_3]$
- $P[p_1 \cap s_2]$
- $P[p_1 \cap s_1]$
- $P[p_2 \cap s_3]$
- $P[p_2 \cap s_2]$
- $P[p_2 \cap s_1]$
- $P[(p_1 \cap s_1) \cup (p_2 \cap s_2)]$
- $P[(p_1 \cap s_2) \cup (p_2 \cap s_1)]$
- $P[(p_1 \cap s_3) \cup (p_2 \cap s_3)]$

More probability examples

At a restaurant, customers are allowed to have (1) two choices for appetizers: soup s or juice j; (2) three for the main course: meat m , fish f, or a vegetable dish v; and (3) two for dessert: ice cream i or cake c.

Assume that the owner of restaurant has observed that 80% of his customers choose the soup for an appetizer and 20% choose juice. Of those who choose soup, 50% choose meat, 30% choose fish, and 20% choose the vegetable dish. Of those who choose juice for an appetizer, 30% choose meat, 40% percent choose fish, and 30% choose the vegetable dish. Finally, customers are just as likely to choose ice cream as they are to choose cake.

Find $P[s \cap m]$, $P[s \cap f]$, $P[s \cap v]$, $P[s \cap m \cap i]$, $P[s \cap f \cap c]$, $P[j \cap m]$, $P[j \cap f]$, $P[j \cap v]$, $P[j \cap m \cap i]$, $P[j \cap f \cap i]$, $P[(s \cap m \cap i) \cup (j \cap v \cap c)]$

More probability examples

A card is chosen at random from a standard deck of 52 playing cards. Without replacing it, a second card is chosen. Let A represent the event that the first card chosen is a queen, and B represents the event that the second card chosen is a jack? Are A and B independent? What is $P[A]$ and $P[B|A]$, and $P[A \cap B]$?

Permutations and Combinations

Definition

Permutations list all possible ways of *ordering* something.
There are three types of permutations:

- Permutation with repetition
- Permutation without repetition of:
  - n out of n elements
  - k out of n elements

Permutations with repetition

Definition

Given a set of n elements, the permutations with repetition are different groups formed by the r elements of a subset such that: ***the order of elements matters, and elements can be repeated***. We denote this as $PR(n,r) = n^r$

Example

The possible ways of choosing a PIN of 4 digest from the set $\{0,1,2,3,4,5,6,7,8,9\}$ are 10^4.

Permutations without repetition

Definition

The permutation of n out of n elements without repetition, is the number of possible ways in which n distinct elements can be arranged. We denote it as $P^n = n!$

Example

There are six permutations of the set $\{1, 2, 3\}$, namely (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), and (3,2,1).
$P^3 = 3! = 3 \times 2 \times 1 = 6$

Permutations without repetition: r out of n

Definition

The permutation of r elements out of n without repetition is:

$$P_r^n = \frac{n!}{(n-r)!}$$

Notice that

$$P_n^n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n! = P^n$$

Example

The ways in which 3 out 5 runners can win 1^{st}, 2^{nd} and 3^{rd} place prizes are $P_5^3 = \frac{5!}{(5-3)!} = 60$

Combinations

Definition

A ***combination*** is the number of ways of arranging r elements out of a larger group of n elements, where (unlike permutations) order does not matter and repetition is not allowed.

$$C_r^n = \frac{n!}{r!(n-r)!}$$

Example

In how many ways can three student-council members be elected from five candidates?
Notice that order doesn't matter here. So, we use

$$C_3^5 = \frac{5!}{3!(5-2)!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1)(2 \times 1)} = \frac{5 \times 4}{2 \times 1} = 10$$

Permutation and Combination Examples

Compute the following:

- C_4^8
- C_2^5
- C_1^4
- C_6^6
- P_2^3
- P_4^6
- P_1^4
- P_3^3